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THE THEORY OF RESONANCE INTERACTION OF TOLLMIEN–SCHLICHTING WAVES

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The resonance interaction of eigenoscillations of a boundary layer is treated by the method of matched asymptotic expansions. It is well known (see for example, [1]) that this is the weakest nonlinear effect in amplitude, following from the linear stages of disturbance evolution and playing an important role in the transition from laminar to turbulent boundary layer. The theoretical study of the effect started with [2-4], and was later extended by many authors [5-8].

In the present study the weakly nonlinear evolutionary equations are derived within the limit of large Reynolds numbers, and the resonance interaction is not assumed ahead of time, but is derived directly from the equations.

The disturbance evolution is treated within the free interaction theory, i.e., one formally uses as original equations the three-dimensional nonstationary boundary layer equations with self-induced pressure, controlling the flow in the boundary region of the boundary layer. Three-wave resonance has already been investigated within this statement of the problem in the high-frequency limit [8], but without including the effect of the critical layer, which, as shown below, plays an important role. This is related to more marked features in a three-dimensional critical layer, while Smith and Stewart [8] obviously based their conclusion concerning "passivity" of the critical layer on investigation results for the two-dimensional case.

The discussion is divided into two parts: in the first we derive the evolution equations by the method of matched asymptotic expansions, and in the second these equations are solved for problems without initial conditions, and the results obtained are briefly discussed.

1. The starting equations consist of the three-layer scheme. The detailed derivation and characteristic orders of magnitude are given, for example, in [9], therefore we do not

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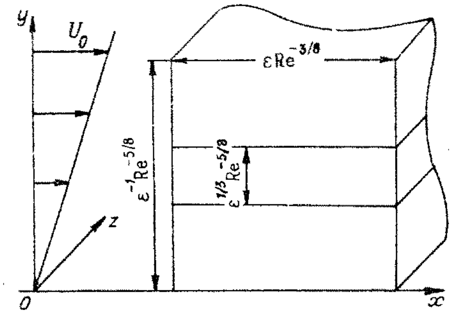

Fig. 1

dwell on them. We write down the basic equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0; \tag{1.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial p}{\partial x}=\frac{\partial^2 u}{\partial y^2}; \tag{1.2}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial p}{\partial z}=\frac{\partial^2 w}{\partial y^2}; \tag{1.3}$$

$$u=w=0 \text{ for } y=0; \tag{1.4}$$

$$v=0 \text{ for } y=0; \tag{1.5}$$

$$u-y=F(x, z, t)+\ldots \text{ for } y\to\infty; \tag{1.6}$$

$$p(x, z, t)=-\frac{1}{2\pi}\int\int_{-\infty}^{+\infty}\frac{\partial^2 F(\xi, \zeta, t)}{\partial \xi^2}\frac{d\xi\, d\zeta}{[(x-\xi)^2+(z-\zeta)^2]^{1/2}}. \tag{1.7}$$

Here u, v, w, and p are, respectively, the velocity components along the Ox, Oy, Oz axes and the pressure (see Fig. 1). According to [10, 11], within the linear statement this system describes Tollmien–Schlichting waves in the vicinity of the lower branch of the neutral curve. Since nonlinear effects are usually observed below the flow, from the point of view of stability loss (which is equivalent to a frequency increase of eigenoscillations in the scale of free interaction theory) it is sufficient to consider the high-frequency limit of the problem. In this case the dispersion relation, relating the components of the wave vector (α, β) with the frequency ω of eigenoscillations, acquires the simple form

$$\alpha(\alpha^2+\beta^2)^{1/2}=\omega+\ldots$$

By direct substitution it can be verified that the triplet of waves with wave vectors

$$(\alpha, \beta),\quad \left(\frac{\alpha-\sqrt{3}\beta}{2}, \frac{\alpha+\sqrt{3}\beta}{2}\right),\quad \left(\frac{\alpha+\sqrt{3}\beta}{2}, \frac{\alpha-\sqrt{3}\beta}{2}\right)$$

consists of a resonance triad for any real α, β, while the phase velocities of all three waves coincide. For this reason it is convenient to use as small parameter ε the reciprocal of the phase velocity c in the triad:

$$\varepsilon=c^{-1}.$$

It follows from the dispersion relation that the spatial disturbance scale is of order $O(\varepsilon)$, and the characteristic time scales as $O(\varepsilon^2)$. Substituting these estimates into system (1.1)-(1.7), one obtains the transverse size $O(\varepsilon^{-1})$. Since the phase velocity is fixed, the critical layer becomes manifested. Its thickness $O(\varepsilon^{1/3})$ is selected in such a manner that the effect of viscosity becomes substantial in this case. By similar considerations one estimates the thickness of the Stokes layer $O(\varepsilon)$. Within the linear approximation the high-frequency oscillations are primarily neutral, therefore their growth can be determined by nonlinear effects even for low amplitudes. In the present study the order of magnitude of "slow" time, for which the disturbance growth due to nonlinear interactions is substantial, is determined as $O(\varepsilon^{2/3})$ from the nonstationarity condition of the equations of motion in the critical layer. In this case the earlier suggested procedure of deriving weakly nonlinear evolution equations [3, 4], assuming uniform smallness of nonlinear corrections in the whole flow region, cannot be applied directly, and requires a special investigation.

The considerations given lead to the following structure of scaled independent variables:

$$t = (\varepsilon^2 t_0,\ \varepsilon^{2/3} t_1),\ x - ct = \varepsilon X,\ z = \varepsilon Z,$$
$$y = (\varepsilon Y_0,\ \varepsilon^{-1} Y_1,\ \varepsilon^{-1} + \varepsilon^{1/3} Y_2). \tag{1.8}$$

The order of magnitude of slow time and the requirement of nonlinear interaction determine the characteristic pressure amplitude as $O(\varepsilon^{10/3})$. To justify this estimate it is necessary to know the form of the solution in the vicinity of the critical layer, which will be determined below. The solution is represented in the form of an asymptotic power series in two small parameters: $\varepsilon^{4/3}$ and ε^2, corresponding to the contributions of the critical and boundary layers. For the pressure we have

$$p(X, Z, t_1) = \varepsilon^{10/3} \frac{\partial}{\partial X} (R_1 + \varepsilon^{4/3} R_2 + \varepsilon^2 R_3 + \varepsilon^{8/3} R_4 + \ldots).$$

Here it has been taken into account that the phase velocity is primarily constant. Therefore, the disturbance depends on "fast" time only through X. For the remaining quantities the expansions depend on which layer they are related to.

We investigate initially the region $Y_1 = O(1)$, restricted by the first three approximations:

$$u = \varepsilon^{-1} Y_1 + \varepsilon^{13/3}(u_1 + \varepsilon^{4/3} u_2 + \varepsilon^2 u_3 + \ldots),$$
$$v = \varepsilon^{7/3}(v_1 + \varepsilon^{4/3} v_2 + \varepsilon^2 v_3 + \ldots),$$
$$w = \varepsilon^{13/3}(w_1 + \varepsilon^{4/3} w_2 + \varepsilon^2 w_3 + \ldots).$$

The equations describing the flow in this region are linear. For $Y_1 = 0$ the nonflow conditions are satisfied in the first and second approximations, while the value of the vertical velocity component must be determined within the third approximation from the matching condition with the solution in a viscous sublayer $Y_0 = O(1)$. The expressions for the first two approximations are written down explicitly

$$\frac{\partial u_1}{\partial X} = \frac{1}{(Y_1 - 1)} \frac{\partial^2 R_1}{\partial Z^2} + \frac{\partial U_0^{\pm}(X, Z, t_1)}{\partial X}, \quad w_1 = -\frac{1}{(Y_1 - 1)} \frac{\partial R_1}{\partial Z},$$
$$v_1 = -\Delta R_1 - (Y_1 - 1) \frac{\partial U_0^{\pm}}{\partial X},$$

$$\frac{\partial^2 u_2}{\partial X^2} = -\frac{1}{(Y_1 - 1)^2} \frac{\partial^3 R_1}{\partial Z^2 \partial t_1} + \frac{1}{(Y_1 - 1)} \frac{\partial^3 R_2}{\partial Z^2 \partial X} + \frac{\partial^2 U_1^{\pm}(X, Z, t_1)}{\partial X^2},$$
$$\frac{\partial w_2}{\partial X} = \frac{1}{(Y_1 - 1)^2} \frac{\partial^2 R_1}{\partial Z \partial t_1} - \frac{1}{(Y_1 - 1)} \frac{\partial R_2}{\partial Z}, \quad v_2 = -\Delta R_2 - \frac{\partial U_0^{\pm}}{\partial t_1} - (Y_1 - 1) \frac{\partial U_1^{\pm}}{\partial X},$$

where $\Delta = \partial_X^2 + \partial_Y^2$, and the superscripts +, − correspond to $Y_1 > 1$, $Y_1 < 1$. These solutions are valid for arbitrary functions $U_0^{\pm}(X, Z, t_1)$, $U_1^{\pm}(X, Z, t_1)$. From the conditions at the wall (1.5) and at the exterior boundary (1.7) it follows that

$$\frac{\partial U_0^-}{\partial X} = \Delta R_1, \quad R_1 = -\frac{1}{2\pi} \int\int_{-\infty}^{+\infty} \frac{\partial U_0^+(\xi, \zeta, t_1)}{\partial \xi} \frac{d\xi\, d\zeta}{[(X - \xi)^2 + (Z - \zeta)^2]^{1/2}}; \tag{1.9}$$

$$\frac{\partial U_1^-}{\partial X} = \Delta R_2 + \frac{\partial U_0^-}{\partial t_1}, \quad R_2 = -\frac{1}{2\pi} \int\int_{-\infty}^{+\infty} \frac{\partial U_1^+(\xi, \zeta, t_1)}{\partial \xi} \frac{d\xi\, d\zeta}{[(X-\xi)^2 + (Z-\zeta)^2]^{1/2}}. \tag{1.10}$$

Relations (1.9), (1.10) are insufficient for unique determination of the functions $U_0^{\pm}$, $U_1^{\pm}$. The continuity condition must not be used for $Y_1 = 1$, since the solution is singular in the critical layer. Establishing a unique relationship between the solutions above and below the critical layer is possible only following detailed treatment of the latter. Within the third approximation it is sufficient to restrict the discussion to terms with lowest powers in $(Y_1 - 1)$:

$$\frac{\partial u_3}{\partial X} = \frac{1}{(Y_1 - 1)} \frac{\partial^2 R_3}{\partial Z^2} + \ldots, \quad w_3 = -\frac{1}{(Y_1 - 1)} \frac{\partial R_3}{\partial Z} + \ldots, \quad v_3 = -\Delta R_3 + \ldots$$

Besides the appearance of singularities in the critical layer, the solutions constructed do not satisfy the adhesion condition (1.4). To satisfy the adhesion condition it is necessary to construct a solution in the Stokes layer $Y_0 = O(1)$; however, it does not primarily affect the pressure distribution and is, therefore, not considered. Within the linear approximation the presence of a Stokes layer leads to deviation from neutrality of the

Tollmien–Schlichting waves considered, while the variable of growth increment is of the order of O(1) (see, for exmaple, [8]). Since it is assumed in the present study that the time of nonlinear interaction is of order $O(\varepsilon^{2/3})$ [with the corresponding increment being $O(\varepsilon^{-2/3})$], the effect of the Stokes layer can indeed be neglected.

Starting from the shape of the solution for $Y_1 = O(1)$, the asymptotic expansion of the velocity in the critical layer can be written down as:

$$u=\varepsilon^{-1}+\varepsilon^{1/3}Y_2+\varepsilon^{3}(\widehat{u}_1+\varepsilon^{4/3}\widehat{u}_2+\varepsilon^{2}\widehat{u}_3+\varepsilon^{8/3}\widehat{u}_4+\ldots),$$
$$v=\varepsilon^{7/3}(-\Delta R_1+\varepsilon^{4/3}\widehat{v}_2+\varepsilon^{2}(-\Delta R_3)+\varepsilon^{8/3}\widehat{v}_4+\ldots),$$
$$w=\varepsilon^{3}(\widehat{w}_1+\varepsilon^{4/3}\widehat{w}_2+\varepsilon^{2}\widehat{w}_3+\varepsilon^{8/3}\widehat{w}_4+\ldots).$$

Substituting these expressions into (1.1)-(1.7), the following system of equations is obtained for the first approximation:

$$\frac{\partial\widehat{u}_1}{\partial t_1}+Y_2\frac{\partial\widehat{u}_1}{\partial X}-\frac{\partial^2 R_1}{\partial Z^2}=\frac{\partial^2\widehat{u}_1}{\partial Y_2^2},\quad \frac{\partial\widehat{w}_1}{\partial t_1}+Y_2\frac{\partial\widehat{w}_1}{\partial X}+\frac{\partial^2 R_1}{\partial Z\,\partial X}=\frac{\partial^2\widehat{w}_1}{\partial Y_2^2},$$
$$\frac{\partial\widehat{u}_1}{\partial X}+\frac{\partial\widehat{w}_1}{\partial Z}=0,\quad (\widehat{u}_1,\widehat{w}_1)\to 0 \text{ for } Y_2\to\infty. \tag{1.11}$$

The problem (1.11) has a solution for any pressure distribution R_1. Introducing a new function Ψ, such that

$$\frac{\partial\Psi}{\partial Z}=-\widehat{u}_1,\quad \frac{\partial\Psi}{\partial X}=\widehat{w}_1,$$

the problem (1.11) can be written down in compact form

$$\frac{\partial\Psi}{\partial t_1}+Y_2\frac{\partial\Psi}{\partial X}+\frac{\partial R_1}{\partial Z}=\frac{\partial^2\Psi}{\partial Y_2^2},\quad \Psi\to 0 \text{ for } Y_2\to\infty. \tag{1.12}$$

For the second approximation the equations acquire the form

$$\frac{\partial\widehat{u}_2}{\partial t_1}+Y_2\frac{\partial\widehat{u}_2}{\partial X}+\widehat{v}_2+\frac{\partial^2 R_2}{\partial X^2}=\frac{\partial^2\widehat{u}_2}{\partial Y_2^2},$$
$$\frac{\partial\widehat{w}_2}{\partial t_1}+Y_2\frac{\partial\widehat{w}_2}{\partial X}+\frac{\partial^2 R_2}{\partial Z\,\partial X}=\frac{\partial^2\widehat{w}_2}{\partial Y_2^2},$$
$$\frac{\partial\widehat{u}_2}{\partial X}+\frac{\partial\widehat{w}_2}{\partial Z}+\frac{\partial\widehat{v}_2}{\partial Y_2}=0,\quad \widehat{u}_2\to U_0^{\pm},\quad \widehat{w}_2\to 0 \text{ for } Y_2\to\infty.$$

The given system is solved with the following condition

$$U_0^{+}=U_0^{-},$$

which, with account of (1.9), closes the problem for the pressure in the principal approximation

$$R_1=-\frac{1}{2\pi}\int\!\!\int_{-\infty}^{+\infty}\Delta R_1(\xi,\zeta,t_1)\frac{d\xi\,d\zeta}{[(X-\xi)^2+(Z-\zeta)^2]^{1/2}}. \tag{1.13}$$

The given integrodifferential equation has the nontrivial solution

$$R_1=\sum_{\varphi\in\{\varphi_j\}}A(t_1,\varphi)\exp[i(X\cos\varphi+Z\sin\varphi)],\quad |\{\varphi_j\}|<\infty, \tag{1.14}$$

which is represented in the form of a superposition of eigenfunctions

$$\exp[i(X\cos\varphi+Z\sin\varphi)],$$

corresponding to Tollmien–Schlichting waves propagating under different angles to the direction of the fundamental flow. The evolution law of the amplitude A is determined from the solvability conditions of higher approximations. For the third approximation the problem is similar to (1.11), and is solved for any amplitude distribution A. The wave interaction appears in the fourth approximation, in which corrections resulting from the flow nonlinearity appear for the first time. The equations governing the fourth approximation functions are

$$\frac{\partial \hat{u}_4}{\partial t_1} + Y_2 \frac{\partial \hat{u}_4}{\partial X} + \hat{v}_4 + \frac{\partial^2 R_4}{\partial X^2} = \frac{\partial^2 \hat{u}_4}{\partial Y_2^2} - \hat{u}_1 \frac{\partial \hat{u}_1}{\partial X} + \Delta R_1 \frac{\partial \hat{u}_1}{\partial Y_2} - \hat{w}_1 \frac{\partial \hat{u}_1}{\partial Z},$$

$$\frac{\partial \hat{w}_4}{\partial t_1} + Y_2 \frac{\partial \hat{w}_4}{\partial X} + \frac{\partial^2 R_4}{\partial Z \partial X} = \frac{\partial^2 \hat{w}_4}{\partial Y_2^2} - \hat{u}_1 \frac{\partial \hat{w}_1}{\partial X} + \Delta R_1 \frac{\partial \hat{w}_1}{\partial Y_2} - \hat{w}_1 \frac{\partial \hat{w}_1}{\partial Z}, \tag{1.15}$$

$$\frac{\partial \hat{u}_4}{\partial X} + \frac{\partial \hat{w}_4}{\partial Z} + \frac{\partial \hat{v}_4}{\partial Y_2} = 0, \quad \hat{u}_4 \to U_1^{\pm}, \quad \hat{w}_4 \to 0 \quad \text{for } Y_2 \to \infty.$$

We introduce a new function Q, related to the vorticity in the critical layer:

$$Q(X, Z, t_1, Y_2) = \frac{\partial^2 \hat{v}_4}{\partial Y_2^2}.$$

We differentiate the first equation with respect to X, Y_2, the second with respect to Z, Y_2, add them, and with account of the continuity equation the system (1.15) is reduced to a problem for the function Q

$$\frac{\partial Q}{\partial t_1} + Y_2 \frac{\partial Q}{\partial X} + q(X, Z, t_1, Y_2) = \frac{\partial^2 Q}{\partial Y_2^2}, \quad Q \to 0 \ \text{for } Y_2 \to \infty,$$

$$q = 2 \frac{\partial}{\partial Y_2} \left[\frac{\partial^2 \Psi}{\partial Z^2} \frac{\partial^2 \Psi}{\partial X^2} - \left(\frac{\partial^2 \Psi}{\partial Z \partial X} \right)^2 \right] - \left(\frac{\partial}{\partial Y_2} \right)^2 \left[\frac{\partial \Psi}{\partial Z} \Delta \frac{\partial R_1}{\partial X} - \frac{\partial \Psi}{\partial X} \Delta \frac{\partial R_1}{\partial Z} \right]; \tag{1.16}$$

$$\int_{-\infty}^{+\infty} Q \, dY_2 = \frac{\partial}{\partial X} (U_1^+ - U_1^-). \tag{1.17}$$

From (1.9), (1.10), and (1.17) one obtains the problem for the second approximation of the pressure function

$$R_2 = -\frac{1}{2\pi} \iint_{-\infty}^{+\infty} \frac{d\xi \, d\zeta}{[(X-\xi)^2 + (Z-\zeta)^2]^{1/2}} \left[\Delta R_2 + \left(\frac{\partial}{\partial X} \right)^{-1} \Delta \frac{\partial R_1}{\partial t_1} + \int_{-\infty}^{+\infty} Q \, dY_2 \right]. \tag{1.18}$$

Equation (1.18) is similar to Eq. (1.13) for the principal approximation, but containing a right-hand side, and consequently it is solved under the orthogonality condition of the right-hand side to all eigenfunctions of problem (1.13):

$$\iint_{-\infty}^{+\infty} \left[\left(\frac{\partial}{\partial X} \right)^{-1} \Delta \frac{\partial R_1}{\partial t_1} + \int_{-\infty}^{+\infty} Q \, dY_2 \right] \exp[i(X \cos\varphi + Z \sin\varphi)] \, dX \, dZ = 0. \tag{1.19}$$

The set (1.12), (1.16), (1.19) makes it possible to determine the time evolution of the amplitude $A(t_1, \varphi)$ from (1.14). We consider the given system in more detail.

2. At the start we note that, generally speaking, this system must be supplemented by initial conditions for Ψ_0, Q_0, A_0 at some initial moment of time t_1^0. We are interested in the behavior of the solution at long times, when the initial conditions have been "forgotten." Formally this is achieved for $t_1^0 \to -\infty$. The solutions for the functions Ψ, Q can then be sought in the form of expansions in the eigenoscillations

$$\Psi = \sum_{\varphi \in \{\varphi_j\}} B(t_1, Y_2, \varphi) \exp[i(X \cos\varphi + Z \sin\varphi)]; \tag{2.1}$$

$$Q = \sum_{\varphi \in \{\varphi_j\}} C(t_1, Y_2, \varphi) \exp[i(X \cos\varphi + Z \sin\varphi)] + Q_1. \tag{2.2}$$

The function C in the representation (2.2) was selected in such a manner that the residual Q_1 satisfied identically the orthogonality conditions (1.19). We thus obtain the transition from the system (1.12), (1.16), (1.19) to the equations for the amplitudes A, B, C (all subscripts are omitted)

$$\frac{\partial B}{\partial t} + iY \cos\varphi B + i \sin\varphi A = \frac{\partial^2 B}{\partial Y^2}; \tag{2.3}$$

$$\frac{\partial C}{\partial t} + iY \cos\varphi C + \frac{3}{2} \frac{\partial (B_+ B_-)}{\partial Y} + \frac{\sqrt{3}}{2} \frac{\partial^2 (A_+ B_- - B_+ A_-)}{\partial Y^2} = \frac{\partial^2 C}{\partial Y^2}; \tag{2.4}$$

$$-\frac{\partial A}{\partial t} + i \cos\varphi \int_{-\infty}^{+\infty} C \, dY = 0, \quad C, B \to 0 \ \text{for } Y \to \infty, \tag{2.5}$$

where $A_\pm = A\left(t, \varphi \pm \frac{\pi}{3}\right)$; $B_\pm = B\left(t, Y, \varphi \pm \frac{\pi}{3}\right)$. As a result $A(t, \varphi)$ depends only on $A(t, \varphi \pm \pi/3)$; consequently, six waves are mutually dependent (for any fixed φ)

$$A\left(t, \varphi + \frac{k\pi}{3}\right), \quad k = 0, 1, \ldots, 5.$$

Taking into account that the pressure is a real function, i.e., $A(\varphi+\pi) = A^*(\varphi)$, the number of related components is three. And the general solution of system (2.3)-(2.5) is decomposed into separate triads not interacting with each other.

Consider the isolated triad

$$A(t, \varphi_0), \; A\left(t, \varphi_0 \pm \frac{\pi}{3}\right).$$

Without loss of generality one can put $|\varphi_0| < \frac{\pi}{6}$. We note that the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} + iY\cos\varphi u = \frac{\partial^2 u}{\partial Y^2}, \quad u|_{t=0} = \delta(Y - Y')$$

is

$$u = \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{(Y-Y')^2}{4t} - \frac{i}{2}\cos\varphi t(Y+Y') - \frac{t^3\cos^2\varphi}{12}\right].$$

With its help one can find the solution of Eqs. (2.3), (2.4):

$$B = -i\sin\varphi \int_{-\infty}^{t} A(t', \varphi) \exp\left[-\frac{1}{3}\cos^2\varphi(t-t')^3 - i\cos\varphi(t-t')Y\right]dt',$$

$$C = -\int_{-\infty}^{t} \frac{dt'}{\sqrt{4\pi(t-t')}} \exp\left[-\frac{1}{3}\cos^2\varphi(t-t')^3\right] \int_{-\infty}^{+\infty} \exp\left[-\frac{(Y-Y')^2}{4t} - \frac{i}{2}\cos\varphi t(Y+Y')\right] q(t', Y', \varphi)\, dY'.$$

An explicit expression for

$$q = \frac{3}{2}\frac{\partial(B_+B_-)}{\partial Y} + \frac{\sqrt{3}}{2}\frac{\partial^2(A_+B_- - B_+A_-)}{\partial Y^2}$$

is obtained by substituting the solution for B. Integrating then the expression for C across the critical layer, from (2.5) we find the integral sought, which is required to construct the evolution equations:

$$\begin{aligned}\int_{-\infty}^{+\infty} C\,dY = &\int_{-\infty}^{t} dt' \int_{-\infty}^{t'}\int_{-\infty}^{t'} dt''dt''' A\left(t'', \varphi + \frac{\pi}{3}\right) A\left(t''', \varphi - \frac{\pi}{3}\right) \times \\ &\times K_0(\varphi, t, t', t'', t''')\, 2\pi\delta\left(t\cos\varphi - t''\cos\left(\varphi + \frac{\pi}{3}\right) - t'''\cos\left(\varphi - \frac{\pi}{3}\right)\right) + \\ &+ \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' A\left(t', \varphi + \frac{\pi}{3}\right) A\left(t'', \varphi - \frac{\pi}{3}\right) K_1(\varphi, t, t', t'')\, 2\pi\delta\Big(t\cos\varphi - \\ &- t'\cos\left(\varphi + \frac{\pi}{3}\right) - t''\cos\left(\varphi - \frac{\pi}{3}\right)\Big) + \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' A\left(t', \varphi - \frac{\pi}{3}\right) \times \\ &\times A\left(t'', \varphi + \frac{\pi}{3}\right) K_2(\varphi, t, t', t'')\, 2\pi\delta\left(t\cos\varphi - t'\cos\left(\varphi - \frac{\pi}{3}\right) - t''\cos\left(\varphi + \frac{\pi}{3}\right)\right).\end{aligned} \tag{2.6}$$

The expressions for the smooth kernels K_0, K_1, K_2 are

$$\begin{aligned}K_0 = &-\frac{3}{2} i \sin\left(\varphi + \frac{\pi}{3}\right)\sin\left(\varphi - \frac{\pi}{3}\right)\Big(\cos\left(\varphi + \frac{\pi}{3}\right)(t' - t'') + \\ &+ \cos\left(\varphi - \frac{\pi}{3}\right)(t' - t''')\Big)\exp\left[-\frac{1}{3}\cos^2\varphi(t-t')^3 - \frac{1}{3}\cos^2\left(\varphi + \frac{\pi}{3}\right)(t'-t'')^3 - \frac{1}{3}\cos^2\left(\varphi - \frac{\pi}{3}\right)(t'-t''')^3\right],\end{aligned}$$

$$K_1 = \frac{\sqrt{3}}{2} i \cos^2\left(\varphi - \frac{\pi}{3}\right)\sin\left(\varphi - \frac{\pi}{3}\right)(t'-t'')^2 \exp\left[-\frac{1}{3}\cos^2\varphi\,(t-t')^3 - \frac{1}{3}\cos^2\left(\varphi - \frac{\pi}{3}\right)(t'-t'')^3\right],$$

$$K_2 = \frac{\sqrt{3}}{2} i \cos^2\left(\varphi + \frac{\pi}{3}\right) \sin\left(\varphi + \frac{\pi}{3}\right)(t' - t'')^2 \exp\left[-\frac{1}{3}\cos^2\varphi \ (t - t')^3 - \frac{1}{3}\cos^2\left(\varphi + \frac{\pi}{3}\right)(t' - t'')^3\right].$$

For $\varphi = \varphi_0$ (by the condition imposed $|\varphi_0| < \frac{\pi}{6}$) the δ-function in expression (2.6) vanishes almost everywhere, and

$$\int\limits_{-\infty}^{+\infty} C dY = 0.$$

From Eq. (2.5) it then follows directly that

$$A(t, \varphi_0) = A_0 = \text{const}$$

i.e., other waves do not affect the φ_0-component of the triad. As a result the equations for the two remaining components are linearized and, since there is no explicit t dependence in the original equations, their amplitudes are written in the form

$$A\left(t, \varphi_0 + \frac{\pi}{3}\right) = a_+ \exp(\lambda t), \ A\left(t, \varphi_0 - \frac{\pi}{3}\right) = a_- \exp(\lambda^* t). \tag{2.7}$$

In principle, by substituting into the system (2.5), (2.6) one can obtain a dispersion relation for λ and for the eigenvector (a_+, a_-) for any φ_0 in the region ($-\pi/6$, $\pi/6$). For the purpose of qualitative analysis, however, one can confine the discussion to the most important and simplest case $\varphi_0 = 0$, i.e., the excitation of subharmonics of a Tollmien–Schlichting plane wave. For this triad substitution of the solutions in the form (2.7) into Eqs. (2.5), (2.6) gives

$$\lambda a_+ = \frac{3\pi i}{8} A_0 I(\lambda) a_-^*, \quad \lambda^* a_- = -\frac{3\pi i}{8} A_0 I(\lambda^*) a_+^*, \tag{2.8}$$
$$I(\lambda) = \int\limits_0^{+\infty} x^2 \exp\left(-2\lambda x - \frac{1}{6} x^3\right) dx.$$

From the solvability condition of system (2.8) with respect to the vector (a_+, a_-) a relation is obtained for the growth rate λ as a function of the amplitude of the two-dimensional wave A_0:

$$\lambda^2 + \left[\frac{3\pi}{8}\right]^2 |A_0|^2 I^2(\lambda) = 0. \tag{2.9}$$

The given relation makes it possible to determine the condition of subharmonic disturbances for a given amplitude of a two-dimensional wave. We investigate the behavior of the growth rate λ in the limiting cases of large and small amplitudes. For the root with a positive real part we have

for $|A_0| \to 0 \quad \lambda = \pm i(3\pi/2)|A_0| + \pi^2(3/4)^{4/3}\Gamma(1/3)|A_0|^2 + \ldots,$

for $|A_0| \to \infty \quad \lambda = \exp(ik\pi/8)(3\pi/32)^{1/4}|A_0|^{1/4} + \ldots, \ k = \pm 1, \pm 3.$

As for roots with negative real parts, since their solutions are damped they are of no interest for this analysis.

In conclusion we summarize the basic results obtained.

1. For sufficiently low amplitudes the general solution of the problem of nonlinear interaction of eigenoscillations within the statement "without initial conditions" is decomposed into isolated triads, evolving independently of each other.

2. For the triad component, whose direction of propagation is overall near the basic flow direction, there is no effect of other components on each other within the approximations considered, and consequently its behavior depends weakly on the behavior of the other components. This behavior is in agreement with experiments [1], in which the amplitude of two-dimensional Tollmien–Schlichting waves was practically unchanged, even when the amplitude of subharmonics exceeded it by more than twice.

3. The equations describing wave evolution are integrodifferential, so that the local growth rate is determined by the whole preceding history of disturbance evolution. As an example we write down the problem for $\varphi_0 = 0$:

$$\frac{dA_0}{dt} = 0,$$

$$\frac{dA_+}{dt} = \frac{3\pi i}{8} \int\limits_{-\infty}^{t} A_0(t') A_-^*(2t' - t)(t - t')^2 \exp\left(-\frac{1}{6}(t - t')^3\right) dt',$$

$$\frac{dA_-}{dt} = -\frac{3\pi i}{8} \int\limits_{-\infty}^{t} A_0(t') A_+^*(2t' - t)(t - t')^2 \exp\left(-\frac{1}{6}(t - t')^3\right) dt'.$$

Here A_0 is the amplitude of the two-dimensional wave, and $A_{+,-}$ are the amplitudes of waves propagating at angels $\pm\pi/3$ to the fundamental flow. The solution is of the form (2.7) with the growth exponent taken from (2.9).

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